A COUNTEREXAMPLE TO A CONJECTURE ON LINEAR SYSTEMS ON $\mathbb{P}^3$

ANTONIO LAFACE AND LUCA UGAGLIA

Abstract. In his paper [1] Ciliberto proposes a conjecture in order to characterize special linear systems of $\mathbb{P}^n$ through multiple base points. In this note we give a counterexample to this conjecture by showing that there is a substantial difference between the speciality of linear systems on $\mathbb{P}^2$ and those of $\mathbb{P}^3$.

Introduction

Let us take the projective space $\mathbb{P}^n$ and let us consider the linear system of hypersurfaces of degree d having some points of fixed multiplicity. The virtual dimension of such systems is the dimension of the space of degree d polynomials minus the conditions imposed by the multiple points and the expected dimension is the maximum between the virtual one and -1. The systems whose dimension is bigger than the expected one are called *special systems*.

There exists a conjecture due to Harbourne [4] and Hirschowitz [6], characterizing special linear systems on $\mathbb{P}^2$, which has been proved in some special cases [2, 3, 8, 7]. Concerning linear systems on $\mathbb{P}^n$, in [1] Ciliberto gives a conjecture based on the classification of special linear systems through double points. In this note we describe a linear system on $\mathbb{P}^3$ that we found in a list of special systems generated with the help of Singular and which turns out to be a counterexample to that conjecture.

The paper is organized as follows: in Section 1 we fix some notations and state Ciliberto's conjecture, while Section 2 is devoted to the counterexample. In Section 3 we try to explain speciality of some systems by the Riemann-Roch formula, and we conclude the note with an appendix containing some computations.

1. Preliminaries

We start by fixing some notations.

Notation 1.1. Let us denote by $\mathbb{L}_n(d, m_1^{a_1}, \ldots, m_r^{a_r})$ the linear system of hypersurfaces of $\mathbb{P}^n$ of degree d, passing through a_i points with multiplicity m_i, for $i = 1, \ldots, r$. Let $\mathcal{I}_Z$ be the ideal of the zero dimensional scheme of multiple points. We denote by $\mathcal{L}_n(d, m_1^{a_1}, \ldots, m_r^{a_r})$ the sheaf $\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z$.

Given the system $\mathbb{L} = \mathbb{L}_n(d, m_1^{a_1}, \ldots, m_r^{a_r})$, its *virtual dimension* is

$$v(\mathbb{L}) = \binom{d+n}{n} - \sum_{i=1}^{r} a_i \binom{m_i + n - 1}{n} - 1,$$

and the *expected dimension* is

$$e(\mathbb{L}) = \max(v(\mathbb{L}), -1).$$

A linear system will be called *special* if its expected dimension is strictly smaller than the effective one.

Remark 1.2. Throughout the paper, if no confusion arises, we will use sometimes the same letter to denote a linear system and the general divisor in the system.

We recall the following definition, see [1].

Definition 1.3. Let X be a smooth, projective variety of dimension n, let C be a smooth, irreducible curve on X and let $\mathcal{N}_{C|X}$ be the normal bundle of C in X. We will say that C is a negative curve if there is a line bundle $\mathcal{N}$ of negative degree and a surjective map $\mathcal{N}_{C|X} \to \mathcal{N}$. The curve C is called a (-1)-curve of size a, with $1 \leq a \leq n-1$, on X if $C \cong \mathbb{P}^1$ and $\mathcal{N}_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus a} \oplus \mathcal{N}$, where $\mathcal{N}$ has no summands of negative degree.

The main conjecture stated in [1] is the following.

Conjecture 1.4. *Let X be the blow-up of $\mathbb{P}^n$ at general points $p_1, \dots, p_r$ and let $\mathbb{L} = \mathbb{L}_n(d, m_1, \dots, m_r)$ be a linear system with multiple base points at $p_1, \dots, p_r$. Then:*

- (i) *the only negative curves on X are (-1)-curves;*
- (ii) *$\mathbb{L}$ is special if and only if there is a (-1)-curve C on X corresponding to a curve Γ on $\mathbb{P}^n$ containing $p_1 \dots, p_r$ such that the general member $D \in \mathbb{L}$ is singular along Γ;*
- (iii) *if $\mathbb{L}$ is special, let B be the component of the base locus of $\mathbb{L}$ containing Γ according to Bertini's theorem. Then the codimension of B in $\mathbb{P}^n$ is equal to the size of C and B appears multiply in the base locus scheme of $\mathbb{L}$.*

In this note we give a counterexample to points (ii) and (iii) of this conjecture.

2. Counterexample

Let us consider the linear system of surfaces of degree nine with one point of degree six and eight points of degree four in $\mathbb{P}^3$, i.e. the system $\mathbb{L} = \mathbb{L}_3(9, 6, 4^8)$. In this section we are going to study this system, showing in particular that it is special but its general member is not singular along a rational curve.
If we denote by $Q = \mathbb{L}_3(2, 1, 1^8)$ the quadric through the nine simple points, we have the following:

Claim 1. $\mathbb{L}_3(9, 6, 4^8) = Q + \mathbb{L}_3(7, 5, 3^8)$.

If we denote by H_1, H_2 two generators of $\mathrm{Pic}(Q)$, considering the restriction $\mathbb{L}_{|Q}$ we get the system of curves in $|9H_1 + 9H_2|$, with one point of multiplicity 6 and eight points of multiplicity 4. We denote for short this system by $|9H_1 + 9H_2| - 6p_0 - \sum 4p_i$.
Looking the Appendix 4.1, we can see that $|9H_1 + 9H_2| - 6p_0 - \sum 4p_i$ corresponds to the planar system $\mathbb{L}_2(12, 3^2, 4^8)$. This last system can not be (-1)-special (see the Appendix 4.2) and $v(\mathbb{L}_2(12, 3^2, 4^8)) = -2$. Therefore, by [8] we may conclude that it is empty.
In particular, also $\mathbb{L}_{|Q} = \emptyset$, and hence $\mathbb{L}$ must contain Q as a fixed component. By subtracting Q from $\mathbb{L}$ we get $\mathbb{L}_3(7, 5, 3^8)$, which proves our claim.

This means that the free part of $\mathbb{L}$ is contained in $\mathbb{L}_3(7, 5, 3^8)$ which has virtual dimension 4. So $\mathbb{L}$ is a special system.

In order to show that $\mathbb{L}$ gives a counterexample to Conjecture 1.4 we are now going to prove that the general member of $\mathbb{L}$ is singular only along the curve C, intersection of Q and $\mathbb{L}_3(7,5,3^8)$, and that C does not contain rational components.
We can consider C as the restriction $\mathbb{L}_3(7,5,3^8)_{|Q}$. This is equal to $|7H_1+7H_2| - 5p_0 - \sum 3p_i$ on the quadric Q, which corresponds to $\mathbb{L}_2(9,2^2,3^8)$ on $\mathbb{P}^2$. This system is not special of dimension 0 and it does not contain rational components (see Appendix 4.3).

Clearly the curve C is contained in $\mathbb{L}_{\text{sing}}$ (i.e. the singular locus of $\mathbb{L}$). We are going to show that in fact $C = \mathbb{L}_{\text{sing}}$.
First of all, let us denote by $\mathbb{L}_3(7,5,3^8,1_Q)$ the subsystem of $\mathbb{L}_3(7,5,3^8)$ obtained by imposing one general simple point on the quadric. Since $\mathbb{L}_3(7,5,3^8,1_Q)_{|Q} = \emptyset$, Q is a fixed component of this system and the residual part is given by $\mathbb{L}_3(5,4,2^8)$. Now $\mathbb{L}_3(5,4,2^8)_{|Q}$ is the system $|5H_1+5H_2|-4p_0-\sum 2p_i$ which corresponds to the non-special system $\mathbb{L}_2(6,1^2,2^8)$, of dimension 1. Therefore, imposing two general simple points on Q and restricting we get that the system $\mathbb{L}_3(5,4,2^8,1^2_Q)_{|Q}$ is empty, which implies that $\mathbb{L}_3(5,4,2^8,1^2_Q)$ has Q as a fixed component. The residual system $\mathbb{L}_3(3,3,1^8)$ is non-special of dimension 1 (because each surface of this system is a cone over a plane cubic through eight fixed points). This implies that the effective dimension of $\mathbb{L}_3(5,4,2^8)$ can not be greater than 3. Therefore it must be 3 since the virtual dimension is 3. By the same argument one shows that the effective dimension of $\mathbb{L}_3(7,5,3^8)$ is 4.
Observe that $\mathrm{Bs}(\mathbb{L}_3(7,5,3^8)) \subseteq \mathrm{Bs}(2Q+\mathbb{L}_3(3,3,1^8))$ since $2Q+\mathbb{L}_3(3,3,1^8) \subseteq \mathbb{L}_3(7,5,3^8)$. So $\mathrm{Bs}(\mathbb{L}_3(7,5,3^8))$ could have only Q as fixed component, but this is not the case since $\dim \mathbb{L}_3(7,5,3^8) = \dim \mathbb{L}_3(5,4,2^8)+1$. The only curves that may belong to $\mathrm{Bs}(\mathbb{L}_3(7,5,3^8))$ are the genus 2 curve $C = \mathbb{L}_3(7,5,3^8)_{|Q}$ and the nine lines of $\mathrm{Bs}(\mathbb{L}_3(3,3,1^8))$ trough the vertex of the cone and each one of the nine base points of the pencil of plane cubics.
We can then conclude that the singular locus $\mathbb{L}_{\text{sing}}$ consists only of the curve C, since the subsystem $3Q+\mathbb{L}_3(3,3,1^8)$ is not singular along the nine fixed lines.

3. Speciality and Riemann-Roch theorem

Let Z be a zero dimensional scheme of $\mathbb{P}^3$ and $\mathcal{I}_Z$ be its ideal sheaf. We put $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z$ and consider the exact sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{L}) \to H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\mathcal{O}_Z) \to H^1(\mathcal{L}) \to H^1(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^1(\mathcal{O}_Z) \\ &\to H^2(\mathcal{L}) \to H^2(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^2(\mathcal{O}_Z) \to H^3(\mathcal{L}) \to H^3(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^3(\mathcal{O}_Z), \end{aligned}$$

obtained tensoring by $\mathcal{O}_{\mathbb{P}^3}(d)$ the sequence defining Z and taking cohomology. From this sequence we obtain that $h^i(\mathcal{L}) = h^i(\mathcal{O}_{\mathbb{P}^3}(d)) = 0$ for $i=2,3$ since $h^i(\mathcal{O}_Z)=0$ for $i=1,2,3$. We also obtain that the virtual dimension of $\mathbb{L}$, $h^0(\mathcal{O}_{\mathbb{P}^3}(d)) - h^0(\mathcal{O}_z) - 1$ is equal to $h^0(\mathcal{L}) - h^1(\mathcal{L}) - 1$ and hence to $\chi(\mathcal{L}) - 1$.
If $Z = \sum m_i p_i$ is a scheme of fat points, then on the blow-up $X \xrightarrow{\pi} \mathbb{P}^3$ along these points we may consider the divisor $\tilde{L} = \pi^*\mathcal{O}_{\mathbb{P}^3}(d) - \sum m_i E_i$ and the associated sheaf $\tilde{\mathcal{L}} = \mathcal{O}_X(\tilde{L})$. Since $h^i(X,\tilde{\mathcal{L}}) = h^i(\mathbb{P}^3,\mathcal{L})$, the virtual dimension of $\mathbb{L}$ is equal to $\chi(\tilde{\mathcal{L}})-1$. By Riemann-Roch formula (see [5]) for a divisor $\tilde{L}$ on the threefold X,

$$\chi(\tilde{L}) = \frac{\tilde{L}(\tilde{L}-K_X)(2\tilde{L}-K_X) + c_2(X)\cdot\tilde{L}}{12} + \chi(\mathcal{O}_X),$$

we obtain the following formula for the virtual dimension of $\mathbb{L}$:

$$v(\mathbb{L}) = \frac{\tilde{L}(\tilde{L} - K_X)(2\tilde{L} - K_X) + c_2(X) \cdot \tilde{L}}{12}$$

since $\chi(\mathcal{O}_X) = 1$.

If the linear system $\mathbb{L}$ can be written as $F + \mathbb{M}$, where F is the fixed divisor and $\mathbb{M}$ is a free part, then on X we have $|\tilde{L}| = \tilde{F} + |\tilde{M}|$. Therefore the above formula says that

$$v(\tilde{L}) = v(\tilde{F}) + v(\tilde{M}) + \frac{\tilde{F}\tilde{M}(\tilde{L} - K_X)}{2}.$$

Let us suppose that the residual system $\mathbb{M}$ is non-special. The system $\mathbb{L}$ has the same effective dimension as $\mathbb{M}$, while their virtual dimensions differ by $v(\tilde{F}) + \tilde{F}\tilde{M}(\tilde{L} - K_X)/2$. Therefore we can conclude that $\mathbb{L}$ is special if $v(\tilde{F}) + \tilde{F}\tilde{M}(\tilde{L} - K_X)/2$ is smaller than zero.

Example 3.1. For instance, let us consider the system $\mathbb{L} := \mathbb{L}_3(4, 2^9)$. It is special because its virtual dimension is -2 while it is not empty since it is equal to $2Q$, where Q is the quadric through the nine simple points. In this case $F = 2Q$ and $\mathbb{M} = \mathbb{C}$, so $v(F) = -2$ and $\tilde{F}\tilde{M}(\tilde{L} - K_X)/2 = 0$.

Example 3.2. Let us consider now the example we described in the previous section, i.e. the system $\mathbb{L}_3(9, 6, 4^8)$. We have seen that it can be written as $Q + \mathbb{M}$, where Q is the quadric through the nine points, while $\mathbb{M} = \mathbb{L}_3(7, 5, 3^8)$ is the residual free part. The Chow ring $A^*(X)$ (where X is the blow-up of $\mathbb{P}^3$ along the nine simple points) is generated by $\langle H, E_0, E_1, \ldots, E_8\rangle$, where H is the pull-back of the hyperplane divisor of $\mathbb{P}^3$ and the E_i's are the exceptional divisors. The second Chow group $A^2(X)$ is generated by $\langle h, e_0, e_1, \ldots, e_8\rangle$, where $h = H^2$ is the pull-back of a line, while $e_i = -E_i^2$ is the class of a line inside E_i, for $i = 0, 1, \ldots, 8$. Clearly $H \cdot E_i = E_i \cdot E_j = 0$ for $i \neq j$. With this notation we can write:

$$\begin{aligned}
\mathbb{L} &= |9H - 6E_0 - \textstyle\sum 4E_i| \\
\mathbb{M} &= |7H - 5E_0 - \textstyle\sum 3E_i| \\
Q &= 2H - E_0 - \textstyle\sum E_i \\
K_X &= -4H + 2E_0 + \textstyle\sum 2E_i.
\end{aligned}$$

Therefore $Q \cdot \mathbb{M} = 14h - 5e_0 - \sum 3e_i$, $\mathcal{L} - K_X = 13H - 8E_0 - \sum 6E_i$ and hence $Q\tilde{M}(\tilde{L} - K_X)/2 = -1$ (while $v(Q) = 0$), which implies the speciality of $\mathbb{L}$.

4. Appendix

4.1. Linear systems on a quadric.

In order to study linear systems on a quadric Q it may be helpful to transform them into planar systems by mean of a birational transformation $Q \to \mathbb{P}^2$ obtained by blowing up a point and contracting the strict transforms of the two lines through it. Such transformation gives rise to a $1:1$ correspondence between linear systems with one multiple point on the quadric and linear systems with two multiple points on $\mathbb{P}^2$.

In fact, let us consider a linear system $|aH_1 + bH_2| - mp$ (i.e. a system of curves

of kind (a,b) through one point p of multiplicity m). Blowing up at p, one obtains the complete system $|a\pi^*H_1 + b\pi^*H_2 - mE|$ which may be written as $|(a+b-m)(\pi^*H_1+\pi^*H_2-E)-(b-m)(\pi^*H_1-E)-(a-m)(\pi^*H_2-E)|$. Since the divisors $\pi^*H_i - E$ $(i=1,2)$ are (-1)-curves, they may be contracted giving a linear system on $\mathbb{P}^2$ of degree $a+b-m$ through two points of multiplicity $b-m$ and $a-m$ and hence

$$|aH_1 + bH_2| - mp \to \mathbb{L}_2(a+b-m, b-m, a-m).$$

4.2. (-1)**-curves.**

In order to study the speciality of the systems $\mathbb{L}_2(12,3^2,4^8)$, $\mathbb{L}_2(9,2^2,3^8)$ and $\mathbb{L}_2(6,1^2,2^8)$, we need to produce a complete list of all the (-1)-curves of $\mathbb{P}^2$ of kind $\mathbb{L}_2(d,m_1,m_2,m_3,\ldots,m_{10})$ which may have an intersection less than -1 with some of these systems. Clearly it is enough to consider the system $\mathbb{L}_2(12,3^2,4^8)$, whose degree and multiplicities are the biggest. From the condition of being contained twice in this system we deduce the following inequalities: $d \le 6,\ 0 \le m_1, m_2 \le 1$ and $0 \le m_3,\ldots,m_{10} \le 2$. Moreover let us see that $m_3 = \ldots = m_{10} = m$. Otherwise the system would contain twice the compound (-1)-curve given by the union of all the simple (-1)-curves obtained by permuting the points $p_3,\ldots,p_{10}$. In this case the multiplicities of the compound curve at these points would be too big. An explicit calculation shows that the only (-1)-curve of the form $\mathbb{L}_2(d,m_1,m_2,m^8)$ satisfying the preceding conditions is $\mathbb{L}_2(1,1,1,0^8)$, but this has non negative intersection with any of these systems.

4.3. $\mathbb{L}_2(9,2^2,3^8)$ **does not contain rational components.**

Let S be the blow up of $\mathbb{P}^2$ along the ten points and let C be the strict transform of the curve given by $\mathbb{L} = \mathbb{L}_2(9,2^2,3^8)$. Suppose that there exists an irreducible rational component C_1 of C. Observe that $v(C_1) = 0$ since the system $|C_1|$ has dimension 0 and it is non-special by [8]. Therefore, from $g(C_1) = v(C_1) = 0$, we get that C_1 is a (-1)-curve.
We are going to see that if this is the case, then $C \cdot C_1 = -1$. Let us take the following exact sequence:

$$0 \to \mathcal{O}_S(C - C_1) \to \mathcal{O}_S(C) \to \mathcal{O}_{\mathbb{P}^1}(C \cdot C_1) \to 0.$$

By the subsection above, $h^1(\mathcal{O}_S(C)) = 0$. Let us see that also $h^1(\mathcal{O}_S(C-C_1)) = 0$. Otherwise the system $|C - C_1|$ would be special and in particular, by [8] there would exist a (-1)-curve C_2 such that $C_2 \cdot (C - C_1) \le -2$. Since $\mathbb{L}$ is non-special, $C \cdot C_2 \ge -1$ and hence $C_1 \cdot C_2 \ge 1$. This implies that $|C_1 + C_2|$ has dimension at least 1, which is impossible since $C_1 + C_2$ is contained in the fixed locus of $\mathbb{L}$. Since $h^0(\mathcal{O}_S(C-C_1)) = h^0(\mathcal{O}_S(C)) = 1$, the cohomology of the preceding sequence gives $h^0(\mathcal{O}_{\mathbb{P}^1}(C\cdot C_1)) = h^1(\mathcal{O}_{\mathbb{P}^1}(C \cdot C_1)) = 0$, which means that $C \cdot C_1 = -1$ as claimed before.
Arguing as in the previous subsection, we get $|C_1| = \mathbb{L}_2(d,m_1,m_2,m^8)$ with $d \le 9,\ 0 \le m_1, m_2 \le 2,\ 0 \le m \le 3$. An easy computation shows that the only (-1)-curve of this form is $\mathbb{L}_2(1,1,1,0^8)$ and in this case $C \cdot C_1 = 5$.

References

[1] Ciro Ciliberto, in *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, 289–316, Progr. Math., 201, Birkhäuser, Basel, 2001

[2] Ciro Ciliberto and Rick Miranda, *Degenerations of planar linear systems*, J. Reine Angew. Math. **501** (1998), 191–220.

[3] ———, *Linear systems of plane curves with base points of equal multiplicity.*, Trans. Am. Math. Soc. **352** (2000), no. 9, 4037–4050 (English).

[4] Brian Harbourne, *The geometry of rational surfaces and Hilbert functions of points in the plane.* In: Proceedings of the 1984 Vancouver conference in algebraic geometry, 95–111, Amer. Math. Soc., Providence, RI, 1986.

[5] Robin Hartshorne, *Algebraic geometry*, Springer, New York, 1977; MR **57**

[6] André Hirschowitz, *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques*, J. Reine Angew. Math. **397** (1989), 208–213.

[7] Evain Laurent, *La fonction de Hilbert de la réunion de* 4^h *gros points génériques de* $\mathbf{p}^2$ *de même multiplicité*, J. Algebraic Geom. **8** (1999), no. 4, 787–796.

[8] Thierry Mignon, *Systèmes de courbes planes à singularités imposées: le cas des multiplicités inférieures ou égales à quatre*, J. Pure Appl. Algebra **151** (2000), no. 2, 173–195.

Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, 20133 MILANO

E-mail address: `laface@mat.unimi.it, ugaglia@mat.unimi.it`